# ON THE ERRORS OF AND POSSIBLE CORRECTIONS TO CLASSICAL LINEAR SHELL THEORY

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PMM Vol.29, № 4, 1965, pp. 701-715

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(Received April 22, 1965)

The results of classical shell theory (the theory based on the Kirchhoff-Love hypothesis) are compared with results obtained from the asymptotic integration of the three-dimensional equations of the theory of elasticity, and a study is made of how these and other arrors which enter the basic relations of classical theory influence the final result.

The investigation deals with the case when there are no boundary supports as, for example, in the case of a shell whose shape is similar to a closed sphere. The only assumptions made are that the variations in the curvatures of the middle surface are sufficiently smooth, that the effective length of the shell is not overly great and that the sought state of stress is formally obtainable by means of the membrane theory for any self-equilibrated load distribution whose components are sufficiently differentiable.

It is generally assumed [1] that the errors of the classical theory are of the order of $h_*$ (nondimensional thickness). This assumption does not take into account the index of variation $t$; for $t = 0$, the above assumption is genrally true, although some counterexamples can be cited even in this case. It has been shown that, if the basic relations contain the errors that are characteristic of classical theory, then for sufficiently long cylindrical [2, 3] or helicoidal [4, 5] shells, the errors may increase to the order of magnitude of unity.

In the present study, error estimates are given taking into account the influence of the index of variation $t$; it is shown that errors may be essentially reduced to quantities of the order of $h_*^{2-2t}$, and the pertinent stress-strain relations are derived.

**1.** Consider the system of orthogonal coordinated $\alpha$, $\beta$ and $\gamma$ in three-dimensional space, where $\alpha$ and $\beta$ are dimensional parameters along the lines of curvature of the middle surface and $\gamma$ is the distance along the normal to this surface.

The shell under investigation in this coordinate system is bounded by the surfaces $\gamma = \pm h$, where the half-thickness $h$ is assumed to be constant. The Lamé coefficients are represented by $H_\alpha$ and $H_\beta$, while the principal radii of curvature of the middle surface given by $R_\alpha$ and $R_\beta$. These quantities are interrelated by Formulas

$$H_\alpha = A\left(1+\frac{\gamma}{R_\alpha}\right), \qquad H_\beta = B\left(1+\frac{\gamma}{R_\beta}\right)$$

where $A$ and $B$ are the coefficients of the first quadratic form of the middle surface.

Subsequent proofs will also make use of Equation

$$\frac{\partial H_\alpha}{\partial\beta} = \frac{\partial A}{\partial\beta}\left(1+\frac{\gamma}{R_\beta}\right) \qquad (\alpha\beta)$$

which follows from the Codazzi equations. Here, as well as hereinafter, the symbol $(\alpha\beta)$ implies that another equation may be obtained by interchanging $\alpha$ with $\beta$ and $A$ with $B$.

We introduce the constant $R$, which represents some characteristic radius of curvature of the middle surface; the ondimensionalized radii of curvature $r_\alpha$ and $r_\beta$, half-thickness $h_*$ and coordinates $\xi$, $\eta$, $\zeta$ are then given by Formulas

$$\frac{h}{R} = h_*, \quad \frac{R_\alpha}{R} = r_\alpha \quad (\alpha\beta), \quad R\frac{\partial}{\partial\alpha} = k\frac{\partial}{\partial\xi} \quad (\alpha\beta), \quad R\frac{\partial}{\partial\gamma} = h_*^{-1}\frac{\partial}{\partial\zeta} \tag{1.1}$$

(The symbol $(\alpha\beta)$ also applies to the interchange of $\xi$ with $\eta$).

In (1.1), $k$ is a constant defined by Equation

$$h_*^{-t} = k \tag{1.2}$$

where $t$ is a nonnegative rational number (the assumption that $t$ is rational does not restrict generality, since the characteristic radius $R$ which was used in the definition of $h_*$ is only restricted to be within a certain range, but otherwise arbitrary). Now introduce another constant parameter $\lambda$ defined by Equations

$$\lambda^p = k, \qquad t = p/q \qquad (p \text{ and } q \text{ are integers}) \tag{1.3}$$

The three-dimensional equations of elasticity will now be written in the coordinate system described above. Taking into account the transformations in the preceding discussion, these equations take the following form:

equilibrium equations

$$\lambda^{p-q}B\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\frac{\partial\sigma_\alpha}{\partial\xi} + \lambda^{p-q}A\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\frac{\partial\tau_{\alpha\beta}}{\partial\eta} + \tag{1.4}$$

$$+ AB\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\frac{\partial\tau_{\alpha\gamma}}{\partial\zeta} + \lambda^{-q}R\frac{\partial B}{\partial\alpha}\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)(\sigma_\alpha-\sigma_\beta) +$$

$$+ 2\lambda^{-q}R\frac{\partial A}{\partial\beta}\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\tau_{\alpha\beta} + \lambda^{-q}AB\left[\left(\frac{2}{r_\alpha}+\frac{1}{r_\beta}\right)+\lambda^{-q}\frac{3\zeta}{r_\alpha r_\beta}\right]\tau_{\alpha\gamma} = 0 \quad (\alpha\beta)$$

$$\lambda^{p-q}B\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\frac{\partial\tau_{\alpha\gamma}}{\partial\xi} + \lambda^{p-q}A\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\frac{\partial\tau_{\beta\gamma}}{\partial\eta} +$$

$$+ \lambda^{-q}R\frac{\partial B}{\partial\alpha}\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\tau_{\alpha\gamma} + \lambda^{-q}R\frac{\partial A}{\partial\beta}\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\tau_{\beta\gamma} -$$

$$-AB\lambda^{-q}\left[\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\frac{\sigma_\alpha}{r_\alpha}+\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\frac{\sigma_\beta}{r_\beta}\right]+$$ (1.4) cont.

$$+AB\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\frac{\partial\sigma_\gamma}{\partial\zeta}+$$

$$+\lambda^{-q}AB\left[\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\frac{1}{r_\beta}+\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\frac{1}{r_\alpha}\right]\sigma_\gamma=0$$

stress-strain relations

$$\frac{E}{R}\frac{\partial u_\gamma}{\partial\zeta}=\lambda^{-q}\left[\sigma_\gamma-\nu(\sigma_\alpha+\sigma_\beta)\right]$$

$$\frac{E}{R}\left(\lambda^p\frac{1}{A}\frac{\partial u_\alpha}{\partial\xi}+\frac{R}{AB}\frac{\partial A}{\partial\beta}u_\beta+\frac{u_\gamma}{r_\alpha}\right)=\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\left[\sigma_\alpha-\nu(\sigma_\gamma+\sigma_\beta)\right]\quad(\alpha\beta)$$

$$\frac{E}{R}\left[\lambda^p\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\frac{1}{B}\frac{\partial u_\alpha}{\partial\eta}+\lambda^p\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\frac{1}{A}\frac{\partial u_\beta}{\partial\xi}-\right.$$

$$\left.-\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\frac{R}{AB}\frac{\partial A}{\partial\beta}u_\alpha-\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\frac{R}{AB}\frac{\partial B}{\partial\alpha}u_\beta\right]=$$

$$=2(1+\nu)\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\left(1+\lambda^{-q}\frac{\zeta}{r_\beta}\right)\tau_{\alpha\beta}\qquad(1.5)$$

$$\frac{E}{R}\left[\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\frac{\partial u_\alpha}{\partial\zeta}+\lambda^{p-q}\frac{1}{A}\frac{\partial u_\gamma}{\partial\xi}-\lambda^{-q}\frac{u_\alpha}{r_\alpha}\right]=$$

$$=2(1+\nu)\lambda^{-q}\left(1+\lambda^{-q}\frac{\zeta}{r_\alpha}\right)\tau_{\alpha\gamma}\quad(\alpha\beta)$$

The foregoing system of equations must now be integrated subject to the conditions on the surfaces $\gamma=\pm h$, i.e. $\zeta=\pm 1$. We will assume that these surfaces carry some arbitrary load distributions. Then

$$\sigma_\gamma=\pm{}^1/_2Q_\gamma-{}^1/_2m,\qquad\tau_{\alpha\gamma}=\pm{}^1/_2Q_\alpha+{}^1/_2M_\beta\quad(\alpha\beta)\qquad\text{for }\zeta=\pm1\qquad(1.6)$$

The quantities appearing in the preceding equations are related to the respective surface-applied force and moment components $X$, $Y$, $Z$ and $E$, $F$ by the formulas used in classical theory (the signs are determined as in [3])

$$X=Q_\alpha+hM_\beta\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)+\frac{h^2}{R_\alpha R_\beta}Q_\alpha\quad(\alpha\beta)$$

$$-Z=Q_\gamma-hm\left(\frac{1}{R_\alpha}+\frac{1}{R_\beta}\right)+\frac{h^2}{R_\alpha R_\beta}Q_\gamma$$

$$F=-hM_\beta,\qquad E=hM_\alpha\qquad(1.7)$$

**2.** As in Section 6 of the paper [6], we will seek a solution to the system if equations (1.4) and (1.5) in the form of asymptotic series in $\lambda$

$$\sigma_\alpha=\lambda^{q+\rho}\sum_s\lambda^{-s}\sigma_\alpha^{(s)}\ (\alpha\beta),\qquad\tau_{\alpha\gamma}=\lambda^{p+\rho}\sum_s\lambda^{-s}\tau_{\alpha\gamma}^{(s)}\ (\alpha\beta),\qquad u_\alpha=\lambda^{q-p+\rho}\sum_s\lambda^{-s}u_\alpha^{(s)}\ (\alpha\beta)$$

$$\sigma_\gamma=\lambda^{\rho}\sum_s\lambda^{-s}\sigma_\gamma^{(s)},\qquad\tau_{\alpha\beta}=\lambda^{q+\rho}\sum_s\lambda^{-s}\tau_{\alpha\beta}^{(s)},\qquad u_\gamma=\lambda^{q+\rho}\sum_s\lambda^{-s}u_\gamma^{(s)}\qquad(2.1)$$

(for the time being, $\rho$ is arbitrary)

Substituting (2.1) into (1.4) and (1.5) and following the usual procedure of equating coefficients of like powers of $\lambda$ , we obtain the following set of systems of equations for the determination of the coefficients in the series expansion:

$$B\frac{\partial\sigma_\alpha^{(s)}}{\partial\xi}+A\frac{\partial\tau_{\alpha\beta}^{(s)}}{\partial\eta}+AB\frac{\partial\tau_{\alpha\gamma}^{(s)}}{\partial\zeta}+R\frac{\partial B}{\partial\alpha}(\sigma_\alpha^{(s-p)}-\sigma_\beta^{(s-p)})+2R\frac{\partial A}{\partial\beta}\tau_{\alpha\beta}^{(s-p)}+\zeta\frac{B}{r_\beta}\frac{\partial\sigma_\alpha^{(s-q)}}{\partial\xi}+$$
$$+\zeta\frac{A}{r_\alpha}\frac{\partial\tau_{\alpha\beta}^{(s-q)}}{\partial\eta}+\zeta AB\left(\frac{1}{r_\alpha}+\frac{1}{r_\beta}\right)\frac{\partial\tau_{\alpha\gamma}^{(s-q)}}{\partial\zeta}+AB\left(\frac{2}{r_\alpha}+\frac{1}{r_\beta}\right)\tau_{\alpha\gamma}^{(s-q)}+\zeta\frac{R}{r_\alpha}\frac{\partial B}{\partial\alpha}(\sigma_\alpha^{(s-p-q)}-$$
$$-\sigma_\beta^{(s-p-q)})+2\zeta\frac{R}{r_\beta}\frac{\partial A}{\partial\beta}\tau_{\alpha\beta}^{(s-p-q)}+AB\left(\frac{\zeta^2}{r_\alpha r_\beta}\frac{\partial\tau_{\alpha\gamma}^{(s-2q)}}{\partial\zeta}+\frac{3\zeta}{r_\alpha r_\beta}\tau_{\alpha\gamma}^{(s-2q)}\right)=0\quad(\alpha\beta)$$

$$-AB\left(\frac{\sigma_\alpha^{(s)}}{r_\alpha}+\frac{\sigma_\beta^{(s)}}{r_\beta}\right)+AB\frac{\partial\sigma_\gamma^{(s)}}{\partial\zeta}+\left\{B\frac{\partial\tau_{\alpha\gamma}^{(s-q+2p)}}{\partial\xi}+A\frac{\partial\tau_{\beta\gamma}^{(s-q+2p)}}{\partial\eta}\right\}+\qquad(2.2)$$
$$+R\frac{\partial B}{\partial\alpha}\tau_{\alpha\gamma}^{(s-q+p)}+R\frac{\partial A}{\partial\beta}\tau_{\beta\gamma}^{(s-q+p)}-\zeta\frac{AB}{r_\alpha r_\beta}(\sigma_\alpha^{(s-q)}+\sigma_\beta^{(s-q)})+\zeta AB\left(\frac{1}{r_\alpha}+\frac{1}{r_\beta}\right)\frac{\partial\sigma_\gamma^{(s-q)}}{\partial\zeta}+$$
$$+AB\left(\frac{1}{r_\alpha}+\frac{1}{r_\beta}\right)\sigma_\gamma^{(s-q)}+\zeta\left(\frac{B}{r_\beta}\frac{\partial\tau_{\alpha\gamma}^{(s-2q+2p)}}{\partial\xi}+\frac{A}{r_\alpha}\frac{\partial\tau_{\beta\gamma}^{(s-2q+2p)}}{\partial\eta}\right)+$$
$$+R\zeta\left(\frac{1}{r_\alpha}\frac{\partial B}{\partial\alpha}\tau_{\alpha\gamma}^{(s-2q+p)}+\frac{1}{r_\beta}\frac{\partial A}{\partial\beta}\tau_{\beta\gamma}^{(s-2q+p)}\right)+\zeta^2\frac{AB}{r_\alpha r_\beta}\frac{\partial\sigma_\gamma^{(s-2q)}}{\partial\zeta}+2\zeta\frac{AB}{r_\alpha r_\beta}\sigma_\gamma^{(s-2q)}=0$$

$$\frac{E}{R}\frac{\partial u_\gamma^{(s)}}{\partial\zeta}=-\nu(\sigma_\alpha^{(s-q)}+\sigma_\beta^{(s-q)})+\sigma_\gamma^{(s-2q)},\quad\frac{E}{R}\left(\frac{1}{A}\frac{\partial u_\alpha^{(s)}}{\partial\xi}+\frac{R}{AB}\frac{\partial A}{\partial\beta}u_\beta^{(s-p)}+\frac{u_\gamma^{(s)}}{r_\alpha}\right)=$$
$$=\sigma_\alpha^{(s)}-\nu\sigma_\beta^{(s)}-\nu\sigma_\gamma^{(s-q)}+\frac{\zeta}{r_\alpha}(\sigma_\alpha^{(s-q)}-\nu\sigma_\beta^{(s-q)}-\nu\sigma_\gamma^{(s-2q)})\quad(\alpha\beta)$$

$$\frac{E}{R}\left(\frac{1}{B}\frac{\partial u_\alpha^{(s)}}{\partial\eta}+\frac{1}{A}\frac{\partial u_\beta^{(s)}}{\partial\xi}-\frac{R}{AB}\frac{\partial A}{\partial\beta}u_\alpha^{(s-p)}-\frac{R}{AB}\frac{\partial B}{\partial\alpha}u_\beta^{(s-p)}+\right.$$
$$\left.+\frac{\zeta}{r_\alpha}\frac{1}{B}\frac{\partial u_\alpha^{(s-q)}}{\partial\eta}+\frac{\zeta}{r_\beta}\frac{1}{A}\frac{\partial u_\beta^{(s-q)}}{\partial\xi}-\frac{\zeta}{AB}\frac{\partial A}{\partial\beta}\frac{R}{r_\beta}u_\alpha^{(s-p-q)}-\frac{\zeta}{AB}\frac{\partial B}{\partial\alpha}\frac{R}{r_\alpha}u_\beta^{(s-p-q)}\right)=$$
$$=2(1+\nu)\left[\tau_{\alpha\beta}^{(s)}+\zeta\left(\frac{1}{r_\alpha}+\frac{1}{r_\beta}\right)\tau_{\alpha\beta}^{(s-q)}+\frac{\zeta^2}{r_\alpha r_\beta}\tau_{\alpha\beta}^{(s-2q)}\right]$$

$$\frac{E}{R}\left(\frac{\partial u_\alpha^{(s)}}{\partial\zeta}+\frac{1}{A}\frac{\partial u_\gamma^{(s-q+2p)}}{\partial\xi}+\frac{\zeta}{r_\alpha}\frac{\partial u_\alpha^{(s-q)}}{\partial\xi}-\frac{1}{r_\alpha}u_\alpha^{(s-q)}\right)=$$
$$=2(1+\nu)\left(\tau_{\alpha\gamma}^{(s-2q+2p)}+\frac{\zeta}{r_\alpha}\tau_{\alpha\gamma}^{(s-3q+2p)}\right)$$

Here, as well as hereinafter, quantities with negative superscripts are taken equal to zero.

**3.** Investigation of system (2.2) will begin with the case where the index of variation of the sought state of stress is zero, i.e. where we may set

$$p=0,\qquad q=1,\qquad \lambda=h^{-1}.\qquad(3.1)$$

Equations (2.2) relate to quantities of order $s$ ; quantities of lower orders are considered to be known. Integration with respect to $\zeta$ is easily accomplished in this system. Performing the integration and taking note of (3.1), we obtain a solution of the form

$$P_j^{(s)} = \sum_{k=0}^{n} \zeta^k P_{j,k}^{(s)} \qquad (\alpha\beta) \tag{3.2}$$

Here $P_j^{(s)}$ represents a typical coefficient of the series expansions in (2.1), and $n = s$ for $\sigma_\alpha^{(s)}$, $\sigma_\beta^{(s)}$, $\tau_{\alpha\beta}^{(s)}$, $u_\alpha^{(s)}$, $u_\beta^{(s)}$ and $u_\gamma^{(s)}$, while for the remaining cases $n = s + 1$.

The dependence of the unknown quantities on $\zeta$ is explicitly shown in (3.2). The quantities bearing the additional subscript $k$ are functions of $\zeta$ and $\eta$ only.

Substituting (3.2) into (2.2) and requiring, in the resultant equations, that the coefficient of each power of $\zeta$ vanish, we obtain a system of partial differential equations (with respect to $\xi$ and $\eta$) for the quantities bearing the additional subscripts.

$$B\frac{\partial\sigma_{\alpha,k}^{(s)}}{\partial\xi} + A\frac{\partial\tau_{\alpha\beta,k}^{(s)}}{\partial\eta} + R\frac{\partial B}{\partial\alpha}(\sigma_{\alpha,k}^{(s)} - \sigma_{\beta,k}^{(s)}) + 2R\frac{\partial A}{\partial\beta}\tau_{\alpha\beta,k}^{(s)} + AB(k+1)\tau_{\alpha\gamma,k+1}^{(s)} =$$

$$= -\frac{1}{r_\alpha}\left[A\frac{\partial\tau_{\alpha\beta,k-1}^{(s-1)}}{\partial\eta} + R\frac{\partial B}{\partial\alpha}(\sigma_{\alpha,k-1}^{(s-1)} - \sigma_{\beta,k-1}^{(s-1)}) + kAB\tau_{\alpha\gamma,k}^{(s-1)}\right] -$$

$$-\frac{1}{r_\beta}\left[B\frac{\partial\sigma_{\alpha,k-1}^{(s-1)}}{\partial\xi} + 2R\frac{\partial A}{\partial\beta}\tau_{\alpha\beta,k-1}^{(s-1)} + kAB\tau_{\alpha\gamma,k}^{(s-2)}\right] -$$

$$-AB\left(\frac{2}{r_\alpha} + \frac{1}{r_\beta}\right)\tau_{\alpha\gamma,k}^{(s-1)} - AB\frac{k+2}{r_\alpha r_\beta}\tau_{\alpha\gamma,k-1}^{(s-r)} \quad (\alpha\beta) \tag{3.3}$$

$$AB\left[\frac{\sigma_{\alpha,k}^{(s)}}{r_\alpha} + \frac{\sigma_{\beta,k}^{(s)}}{r_\beta} - (k+1)\sigma_{\gamma,k+1}^{(s)}\right] = B\frac{\partial\tau_{\alpha\gamma,k}^{(s-1)}}{\partial\xi} + A\frac{\partial\tau_{\beta\gamma,k}^{(s-1)}}{\partial\eta} + R\frac{\partial B}{\partial\alpha}\tau_{\alpha\gamma,k}^{(s-1)} + R\frac{\partial A}{\partial\beta}\tau_{\beta\gamma,k}^{(s-1)} +$$

$$+ AB\left(\frac{1}{r_\alpha} + \frac{1}{r_\beta}\right)\sigma_{\gamma,k}^{(s-1)} - AB\left[\frac{1}{r_\alpha r_\beta}(\sigma_{\alpha,k-1}^{(s-1)} + \sigma_{\beta,k-1}^{(s-1)}) - k\left(\frac{1}{r_\alpha} + \frac{1}{r_\beta}\right)\sigma_{\gamma,k}^{(s-1)}\right] +$$

$$+\left(\frac{B}{r_\beta}\frac{\partial\tau_{\alpha\gamma,k-1}^{(s-2)}}{\partial\xi} + \frac{A}{r_\alpha}\frac{\partial\tau_{\beta\gamma,k-1}^{(s-2)}}{\partial\eta}\right) + \frac{R}{r_\alpha}\frac{\partial B}{\partial\alpha}\tau_{\alpha\gamma,k-1}^{(s-2)} + \frac{R}{r_\beta}\frac{\partial A}{\partial\beta}\tau_{\beta\gamma,k-1}^{(s-2)} + (k+1)\frac{AB}{r_\alpha r_\beta}\sigma_{\gamma,k-1}^{(s-2)}$$

$$\frac{E}{R}ku_{\gamma,k}^{(s)} = -\nu(\sigma_{\alpha,k-1}^{(s-1)} + \sigma_{\beta,k-1}^{(s-1)}) + \sigma_{\gamma,k-1}^{(s-2)}$$

$$\frac{E}{R}ku_{\alpha,k}^{(s)} = -\frac{E}{R}\left(\frac{1}{A}\frac{\partial u_{\gamma,k-1}^{(s-1)}}{\partial\xi} + \frac{u_{\alpha,k-1}^{(s-1)}}{r_\alpha} - \frac{k}{r_\alpha}u_{\alpha,k}^{(s-1)}\right) +$$

$$+ 2(1+\nu)\left(\tau_{\alpha\gamma,k-1}^{(s-2)} + \frac{1}{r_\alpha}\tau_{\alpha\gamma,k-2}^{(s-3)}\right) \quad (\alpha\beta)$$

$$\frac{E}{R}\left(\frac{1}{A}\frac{\partial u_{\alpha,k}^{(s)}}{\partial\xi} + \frac{R}{AB}\frac{\partial A}{\partial\beta}u_{\beta,k}^{(s)} + \frac{u_{\gamma,k}^{(s)}}{r_\alpha}\right) = \sigma_{\alpha,k}^{(s)} - \nu\sigma_{\beta,k}^{(s)} +$$

$$+ \frac{1}{r_\alpha}(\sigma_{\alpha,k-1}^{(s-1)} - \nu\sigma_{\beta,k-1}^{(s-1)}) - \nu\sigma_{\gamma,k}^{(s-1)} - \frac{\nu}{r_\alpha}\sigma_{\alpha,k-1}^{(s-2)} \quad (\alpha\beta)$$

$$\frac{E}{R}\left(\frac{1}{B}\frac{\partial u_{\alpha,k}^{(s)}}{\partial\eta} + \frac{1}{A}\frac{\partial u_{\beta,k}^{(s)}}{\partial\xi} - \frac{R}{AB}\frac{\partial A}{\partial\beta}u_{\alpha,k}^{(s)} - \frac{R}{AB}\frac{\partial B}{\partial\alpha}u_{\beta,k}^{(s)}\right) +$$

$$+ \frac{E}{R}\left[\frac{1}{r_\alpha}\left(\frac{1}{B}\frac{\partial u_{\alpha,k-1}^{(s-1)}}{\partial\eta} - \frac{R}{AB}\frac{\partial B}{\partial\alpha}u_{\beta,k-1}^{(s-1)}\right) + \frac{1}{r_\beta}\left(\frac{1}{A}\frac{\partial u_{\beta,k-1}^{(s-1)}}{\partial\xi} - \frac{R}{AB}\frac{\partial A}{\partial\beta}u_{\alpha,k-1}^{(s-1)}\right)\right] =$$

$$= 2(1+\nu)\left[\tau_{\alpha\beta,k}^{(s)} + \left(\frac{1}{r_\alpha} + \frac{1}{r_\beta}\right)\tau_{\alpha\beta,k-1}^{(s-1)} + \frac{1}{r_\alpha r_\beta}\tau_{\alpha\beta,k-2}^{(s-2)}\right]$$

In these equations

$$\sigma_{\alpha,l}^{(t)} = \sigma_{\beta,l}^{(t)} = \tau_{\alpha\beta,l}^{(t)} = u_{\alpha,l}^{(t)} = u_{\beta,l}^{(t)} = u_{\gamma,l}^{(t)} \equiv 0, \quad \text{if} \quad t<0 \quad \text{or} \quad l<0 \quad \text{or} \quad t>l$$

$$\tau_{\alpha\gamma,l}^{(t)} = \tau_{\beta\gamma,l}^{(t)} = \sigma_{\gamma,l}^{(t)} \equiv 0, \quad \text{if} \quad t<0 \quad \text{or} \quad l<0 \quad \text{or} \quad t>l+1 \tag{3.4}$$

To Equations (3.3), we must adjoin (1.6), expressing the conditions on the exterior and interior surfaces. Assume that $Q_\alpha$, $Q_\beta$, $Q_\gamma$, $M_\alpha$ $M_\beta$, and $m$, on these surfaces, are independent of $h_*$ ; then we may set $\rho = 0$ in (2.1), and (2.1) will be satisfied, if

$$\sigma_\gamma^{(0)} = \pm {}^1\!/\!_2 Q_\gamma - {}^1\!/\!_2 m, \qquad \tau_{\alpha\gamma}^{(0)} = \pm {}^1\!/\!_2 Q_\alpha + {}^1\!/\!_2 M_\beta \quad (\alpha, \beta)$$

$$\sigma_\gamma^{(j)} = \tau_{\alpha\gamma}^{(j)} = \tau_{\beta\gamma}^{(j)} = 0 \qquad j>0 \qquad \text{for} \quad \zeta = \pm 1,$$

Substitution of (3.2) into these equations yields

$$\tau_{\alpha\gamma,0}^{(0)} = {}^1\!/\!_2 M_\beta \quad (\alpha\beta), \qquad \sigma_{\gamma,0}^{(0)} = {}^1\!/\!_2 m; \quad \tau_{\alpha\gamma,1}^{(0)} = {}^1\!/\!_2 Q_\alpha \quad (\alpha\beta), \qquad \sigma_{\gamma,1}^{(0)} = {}^1\!/\!_2 Q_\gamma$$

$$\tau_{\alpha\gamma,1}^{(1)} = 0 \quad (\alpha\beta), \qquad \sigma_{\gamma,1}^{(1)} = 0 \tag{3.5}$$

$$\tau_{\alpha\gamma,0}^{(1)} + \tau_{\alpha\gamma,2}^{(1)} = 0 \quad (\alpha\beta), \qquad \sigma_{\gamma,0}^{(1)} + \sigma_{\gamma,2}^{(1)} = 0$$

$$\sum_{i=0}^{r} \tau_{\alpha\gamma,2i}^{(j)} = 0 \quad (\alpha\beta), \qquad \sum_{i=0}^{r} \tau_{\alpha\gamma,2i+1}^{(j)} = 0 \quad (\alpha\beta), \quad \sum_{i=0}^{r} \sigma_{\gamma,2i}^{(j)} = 0, \quad \sum_{i=0}^{r} \sigma_{\gamma,2i+1}^{(j)} = 0 \quad (j>1)$$

where $r$ is the integral part of the quantity $\frac{1}{2}j + \frac{1}{2}$ .

As will be seen later on, the combination of equations in (3.3) and (3.5) is sufficient to permit the sequential determination of the coefficients of the series in (2.1) and (3.2).

4. Set $k = 0$ in (3.3). Then, taking into account (3.4), we obtain

$$B\frac{\partial \sigma_{\alpha,0}^{(s)}}{\partial \xi} + A\frac{\partial \tau_{\alpha\beta,0}^{(s)}}{\partial \eta} + R\frac{\partial B}{\partial \alpha}(\sigma_{\alpha,0}^{(s)} - \sigma_{\beta,0}^{(s)}) + 2R\frac{\partial A}{\partial \beta}\tau_{\alpha\beta,0}^{(s)} + ABR_\alpha^{(s)} = 0 \quad (\alpha\beta)$$

$$\frac{\sigma_{\alpha 0}^{(s)}}{r_\alpha} + \frac{\sigma_{\beta,0}^{(s)}}{r_\beta} - R_\gamma^{(s)} = 0$$

$$\frac{E}{R}\left(\frac{1}{A}\frac{\partial u_{\alpha,0}^{(s)}}{\partial \xi} + \frac{R}{AB}\frac{\partial A}{\partial \beta}u_{\beta,0}^{(s)} + \frac{u_{\gamma,0}^{(s)}}{r_\alpha}\right) - \sigma_{\alpha,0}^{(s)} + \nu\sigma_{\beta,0}^{(s)} = P^{(s)} \quad (\alpha\beta) \tag{4.1}$$

$$\frac{E}{R}\left(\frac{1}{B}\frac{\partial u_{\alpha,0}^{(s)}}{\partial \eta} + \frac{1}{A}\frac{\partial u_{\beta,0}^{(s)}}{\partial \xi} - \frac{R}{AB}\frac{\partial A}{\partial \beta}u_{\alpha,0}^{(s)} - \frac{R}{AB}\frac{\partial B}{\partial \alpha}u_{\beta,0}^{(s)}\right) - 2(1+\nu)\tau_{\alpha\beta}^{(s)} = 0$$

where

$$R_\alpha^{(s)} = \tau_{\alpha\gamma,1}^{(s)} + \left(\frac{2}{r_\alpha} + \frac{1}{r_\beta}\right)\tau_{\alpha\gamma,0}^{(s-1)} \quad (\alpha\beta)$$

$$R_\gamma^{(s)} = \sigma_{\gamma,1}^{(s)} + \frac{1}{A}\frac{\partial \tau_{\alpha\gamma,0}^{(s-1)}}{\partial \xi} + \frac{1}{B}\frac{\partial \tau_{\beta\gamma,0}^{(s-1)}}{\partial \eta} + \frac{R}{AB}\frac{\partial B}{\partial \alpha}\tau_{\alpha\gamma,0}^{(s-1)} + $$

$$+ \frac{R}{AB}\frac{\partial A}{\partial \beta}\tau_{\beta\gamma,0}^{(s-1)} + \left(\frac{1}{r_\alpha} + \frac{1}{r_\beta}\right)\sigma_{\gamma,0}^{(s-1)}$$

$$P^{(s)} = -\nu\sigma_{\gamma,0}^{(s-1)} \tag{4.2}$$

(when $k = 0$, the third and fourth equations in (3.3) are reduced to identities, in virtue of (3.4)).

If we consider $R_\alpha^{(s)}$, $R_\beta^{(s)}$, $R_\gamma^{(s)}$ and $P^{(s)}$, to be known, then (4.1) comprises a system of six partial differential equations (with respect to $\xi$

and $\eta$) in the six unknowns $\sigma_{\alpha,0}^{(s)}$, $\sigma_{\beta,0}^{(s)}$, $\tau_{\alpha\beta,0}^{(s)}$, $u_{\alpha,0}^{(s)}$, $u_{\beta,0}^{(s)}$ and $u_{\gamma,0}^{(s)}$, defining the membrane portion of the unknown state of stress, i.e. the portion in which the stresses $\sigma_\alpha$, $\tau_{\alpha\beta}$, $\sigma_\beta$ are constant over the thickness.

For $s = 0$ and $s = 1$, the quantities $R_\alpha^{(s)}$, $R_\beta^{(s)}$, $R_\gamma^{(s)}$ and $P^{(s)}$, may be expressed in terms of the components of the external loading by making use of (3.4), (3.5) and (4.2). Thus,

$$R_\alpha^{(0)} = {}^1/_2 Q_\alpha \quad (\alpha\beta), \qquad R_\gamma^{(0)} = {}^1/_2 Q_\gamma, \qquad P^{(0)} = 0$$
$$R_\alpha^{(1)} = \left(\frac{2}{r_\alpha} + \frac{1}{r_\beta}\right)\frac{M_\beta}{2} \quad (\alpha\beta), \qquad P^{(1)} = \frac{\nu}{2} m \tag{4.3}$$

$$R_\gamma^{(1)} = \frac{1}{2}\left[\frac{1}{A}\frac{\partial M_\beta}{\partial \xi} + \frac{1}{B}\frac{\partial M_\alpha}{\partial \eta} + \frac{R}{AB}\frac{\partial B}{\partial \alpha} M_\beta + \frac{R}{AB}\frac{\partial A}{\partial \beta} M_\alpha - \left(\frac{1}{r_\alpha} + \frac{1}{r_\beta}\right) m\right]$$

For $s > 1$, the quantities $R_\alpha^{(s)}$, $R_\beta^{(s)}$, $R_\gamma^{(s)}$ and $P^{(s)}$ will also be known (if all lower order approximations have been constructed), but the expressions for these quantities will include, in addition to the components of external loading, those coefficients in the series expansions (2.1) whose indices are less than $s$.

The quantities $\sigma_{\alpha,1}^{(s)}$, $\tau_{\alpha\beta,1}^{(s)}$, $\sigma_{\beta,1}^{(s)}$, $u_{\alpha,1}^{(s)}$, $u_{\beta,1}^{(s)}$ and $u_{\gamma,1}^{(s)}$ in the series expansions (2.1) and (3.2) define the pure moment portion of the state of stress, i.e. the part in which the stresses $\sigma_\alpha$, $\tau_{\alpha\beta}$ and $\sigma_\beta$ have a linear antisymmetric variation over the shell thickness. To determine these quantities, it is sufficient to set $k = 1$ in the last four equations of (3.3). We obtain

$$\frac{E}{R} u_{\gamma,1}^{(s)} = -\nu\left(\sigma_{\alpha,0}^{(s-1)} + \sigma_{\beta,0}^{(s-1)}\right) + \sigma_{\gamma,0}^{(s-2)}$$

$$\frac{E}{R} u_{\alpha,1}^{(s)} = -\frac{E}{R}\left(\frac{1}{A}\frac{\partial u_{\gamma,0}^{(s-1)}}{\partial \xi} + \frac{u_{\alpha,0}^{(s-1)}}{r_\alpha} - \frac{u_{\alpha,1}^{(s-1)}}{r_\alpha}\right) + 2(1+\nu)\tau_{\alpha\gamma,0}^{(s-2)}$$

$$\frac{E}{R}\left(\frac{1}{A}\frac{\partial u_{\alpha,1}^{(s)}}{\partial \xi} + \frac{R}{AB}\frac{\partial A}{\partial \beta} u_{\beta,1}^{(s)} + \frac{u_{\gamma,1}^{(s)}}{r_\alpha}\right) =$$
$$= \sigma_{\alpha,1}^{(s)} - \nu\sigma_{\beta,1}^{(s)} + \frac{1}{r_\alpha}\left(\sigma_{\alpha,0}^{(s-1)} - \nu\sigma_{\beta,0}^{(s-1)}\right) - \nu\sigma_{\gamma,1}^{(s-1)} - \frac{\nu}{r_\alpha}\sigma_{\alpha,0}^{(s-2)} \quad (\alpha\beta)$$

$$\frac{E}{R}\left(\frac{1}{B}\frac{\partial u_{\alpha,1}^{(s)}}{\partial \eta} + \frac{1}{A}\frac{\partial u_{\beta,1}^{(s)}}{\partial \xi} - \frac{R}{AB}\frac{\partial A}{\partial \beta} u_{\alpha,1}^{(s)} - \frac{R}{AB}\frac{\partial B}{\partial \alpha} u_{\beta,1}^{(s)}\right) + \tag{4.4}$$
$$+ \frac{E}{R}\left[\frac{1}{r_\alpha}\left(\frac{1}{B}\frac{\partial u_{\alpha,0}^{(s-1)}}{\partial \eta} - \frac{R}{AB}\frac{\partial B}{\partial \alpha} u_{\beta,0}^{(s-1)}\right) + \frac{1}{r_\beta}\left(\frac{1}{A}\frac{\partial u_{\beta,0}^{(s-1)}}{\partial \xi} - \right.\right.$$
$$\left.\left. - \frac{R}{AB}\frac{\partial A}{\partial \beta} u_{\alpha,0}^{(s-1)}\right)\right] = 2(1+\nu)\left[\tau_{\alpha\beta,1}^{(s)} + \left(\frac{1}{r_\alpha} + \frac{1}{r_\beta}\right)\tau_{\alpha\beta,0}^{(s-1)}\right]$$

Whence, considering all lower order approximations up to and including the $(s-1)$th to be known, differentiation and algebraic manipulation yield the expressions for $u_{\gamma,1}^{(s)}$, $u_{\alpha,1}^{(s)}$, $u_{\beta,1}^{(s)}$, $\sigma_{\alpha,1}^{(s)}$, $\sigma_{\beta,1}^{(s)}$ and $\tau_{\alpha\beta,1}^{(s)}$. For $k = 1$, the first two equations in (3.3) yield (also by means of simple operations) $\tau_{\alpha\gamma,2}^{(s)}$, $\tau_{\beta\gamma,2}^{(s)}$ and $\sigma_{\gamma,2}^{(s)}$, which are needed for the determination of $R_\alpha^{(s)}$, $R_\beta^{(s)}$,

$R_\gamma^{(s)}$ and $P^{(s)}$.

5. We will confine ourselves to the first two terms in the series expansions (2.1). Then, in conformance with (3.2), the formulas for the stresses and displacements are

$$\sigma_\alpha = h^{-1} R(\sigma_{\alpha,0}^{(0)} + \frac{h}{R}\sigma_{\alpha,0}^{(1)} + \frac{h}{R}\zeta\sigma_{\alpha,1}^{(1)}) \qquad (\alpha\beta)$$

$$\tau_{\alpha\beta} = h^{-1} R(\tau_{\alpha\beta,0}^{(0)} + \frac{h}{R}\tau_{\alpha\beta,0}^{(1)} + \frac{h}{R}\zeta\tau_{\alpha\beta,1}^{(1)})$$

$$\tau_{\alpha\gamma} = \tau_{\alpha\gamma,0}^{(0)} + \frac{h}{R}\tau_{\alpha\gamma,0}^{(1)} + \zeta(\tau_{\alpha\gamma,1}^{(0)} + \frac{h}{R}\tau_{\alpha\gamma,1}^{(1)}) + \frac{h}{R}\zeta^2\tau_{\alpha\gamma,2}^{(1)} \tag{5.1}$$

$$\sigma_\gamma = \sigma_{\gamma,0}^{(0)} + \frac{h}{R}\sigma_{\gamma,0}^{(1)} + \zeta(\sigma_{\gamma,1}^{(0)} + \frac{h}{R}\sigma_{\gamma,1}^{(1)}) + \frac{h}{R}\zeta^2\sigma_{\gamma,2}^{(1)}$$

$$u_\alpha = h^{-1} R(u_{\alpha,0}^{(0)} + \frac{h}{R}u_{\alpha,0}^{(1)} + \frac{h}{R}\zeta u_{\alpha,1}^{(1)}) \ (\alpha\beta), \quad u_\gamma = h^{-1} R(u_{\gamma,0}^{(0)} + \frac{h}{R}u_{\gamma,0}^{(1)} + \frac{h}{R}\zeta u_{\gamma,1}^{(1)})$$

The quantities in the right-hand sides of these equations may be determined by means of the equations obtained in Section 4. An approximate method for the analysis of shells is thus constructed. The results obtained will now be expressed in terms of classical shell theory.

The stress resultants, moment resultants and displacement components for the middle surface may be written as

$$T_1 = T_1^{(0)} + \frac{h}{R}T_1^{(1)} = \int_{-h}^{+h} \sigma_\alpha \left(1 + \frac{\gamma}{R_\beta}\right) d\gamma \qquad (\alpha\beta)$$

$$S_1 = S_1^{(0)} + \frac{h}{R}S_1^{(1)} = \int_{-h}^{+h} \tau_{\alpha\beta} \left(1 + \frac{\gamma}{R_\beta}\right) d\gamma$$

$$S_2 = S_2^{(0)} + \frac{h}{R}S_2^{(1)} = -\int_{-h}^{+h} \tau_{\alpha\beta} \left(1 + \frac{\gamma}{R_\alpha}\right) d\gamma$$

$$G_1 = -\int_{-h}^{+h} \sigma_\alpha \gamma \left(1 + \frac{\gamma}{R_\beta}\right) d\gamma \qquad (\alpha\beta) \tag{5.2}$$

$$H_1 = \int_{-h}^{+h} \tau_{\alpha\beta}\gamma \left(1 + \frac{\gamma}{R_\beta}\right) d\gamma, \qquad H_2 = -\int_{-h}^{+h} \tau_{\alpha\beta}\gamma \left(1 + \frac{\gamma}{R_\alpha}\right) d\gamma$$

$$u = u^{(0)} + \frac{h}{R}u^{(1)} = Rh^{-1}(u_{\alpha,0}^{(0)} + \frac{h}{R}u_{\alpha,0}^{(1)}) \qquad (\alpha\beta)$$

$$w = w^{(0)} + \frac{h}{R}w^{(1)} = -Rh^{-1}(u_{\gamma,0}^{(0)} + \frac{h}{R}u_{\gamma,0}^{(1)})$$

Here Love's notation has been used [7]. Substituting (5.1) into (5.2), we obtain

$$T_1^{(i)} = 2R\sigma_{\alpha,0}^{(i)} \ (\alpha\beta), \quad S_1 = 2R\tau_{\alpha\beta,0}^{(i)}, \quad S_2 = -2R\tau_{\alpha\beta,0}^{(i)}$$
$$u^{(i)} = Rh^{-1}u_{\alpha,0}^{(i)} \ (\alpha\beta), \quad w^{(i)} = -Rh^{-1}u_{\gamma,0}^{(i)} \qquad (i = 0, 1) \tag{5.3}$$

$$G_1 = -\frac{2h^2}{3}\left(\sigma_{\alpha,\,1}^{(1)} + \frac{\sigma_{\alpha,\,0}^{(0)}}{r_\beta}\right) \qquad (\alpha\beta)$$

$$H_1 = \frac{2h^2}{3}\left(\tau_{\alpha\beta,\,1}^{(1)} + \frac{\tau_{\alpha\beta,\,0}^{(0)}}{r_\beta}\right), \qquad H_2 = -\frac{2h^2}{3}\left(\tau_{\alpha\beta,\,1}^{(1)} + \frac{\tau_{\alpha\beta,\,0}^{(0)}}{r_\alpha}\right) \qquad (5.4)$$

Substitution of (5.3) into (4.1) yields zeroth and first order approximation equations in terms of the tangential stress resultants and displacements (these quantities are designated, respectively, by the indices 0 and 1 ). Utilizing these results and returning from nondimensional to dimensional quantities by means of Formulas (1.1), we obtain

(in all formulas $i = 0, 1$)

$$B\frac{\partial T_1^{(i)}}{\partial\alpha} - A\frac{\partial S_2^{(i)}}{\partial\beta} + \frac{\partial B}{\partial\alpha}(T_1^{(i)} - T_2^{(i)}) + \frac{\partial A}{\partial\beta}(S_1^{(i)} - S_2^{(i)}) + 2ABR_\alpha^{(i)} = 0$$

$$B\frac{\partial S_1^{(i)}}{\partial\alpha} + A\frac{\partial T_2^{(i)}}{\partial\beta} + \frac{\partial B}{\partial\alpha}(S_1^{(i)} - S_2^{(i)}) + \frac{\partial A}{\partial\beta}(T_2^{(i)} - T_1^{(i)}) + 2ABR_\beta^{(i)} = 0$$

$$\frac{T_1^{(i)}}{R_\alpha} + \frac{T_2^{(i)}}{R_\beta} - 2R_\gamma^{(i)} = 0 \qquad (5.5)$$

$$T_1^{(i)} - \nu T_2^{(i)} = 2Eh\varepsilon_1^{(i)} + 2P^{(i)}, \qquad T_2^{(i)} - \nu T_1^{(i)} = 2Eh\varepsilon_2^{(i)} + 2P^{(i)}$$

$$2(1+\nu)S_1^{(i)} = -2(1+\nu)S_2^{(i)} = 2Eh\omega^{(i)} \qquad (5.6)$$

$$\varepsilon_1^{(i)} = \frac{1}{A}\frac{\partial u^{(i)}}{\partial\alpha} + \frac{1}{AB}\frac{\partial A}{\partial\beta}v^{(i)} - \frac{w^{(i)}}{R_\alpha}, \quad \varepsilon_2^{(i)} = \frac{1}{B}\frac{\partial v^{(i)}}{\partial\beta} + \frac{1}{AB}\frac{\partial B}{\partial\alpha}u^{(i)} - \frac{w^{(i)}}{R_\beta}$$

$$\omega^{(i)} = \frac{A}{B}\frac{\partial}{\partial\beta}\frac{u^{(i)}}{A} + \frac{B}{A}\frac{\partial}{\partial\alpha}\frac{v^{(i)}}{B} \qquad (5.7)$$

$R_\alpha^{(0)}, R_\beta^{(0)}, R_\gamma^{(0)}$ and $P^{(0)}$ are given by Formulas (4.3). From these it follows that, when $i = 0$ , Equations (5.5) will coincide with the equations of equilibrium of membrane theory if, in the latter, the components of the exterior surface loading are taken, respectively, as

$$Q_\alpha, \quad Q_\beta, \quad -Q_\gamma \qquad (5.8)$$

When $P^{(i)} = 0$, (5.6) and (5.7) with $i = 0$, coincide with stress-strain relations for the tangential stress resultants.

When $i = 1$ , the meaning of Equations (5.5) to (5.7) is the same as before, but the components of the exterior surface loading are now given, respectively, in accordance with (4.3) as

$$\left(\frac{2}{r_\alpha} + \frac{1}{r_\beta}\right)M_\beta, \qquad \left(\frac{1}{r_\alpha} + \frac{2}{r_\beta}\right)M_\alpha \qquad (5.9)$$

$$\left[\frac{1}{A}\frac{\partial M_\beta}{\partial\xi} + \frac{1}{B}\frac{\partial M_\alpha}{\partial\eta} + \frac{R}{AB}\frac{\partial B}{\partial\alpha}M_\beta + \frac{R}{AB}\frac{\partial A}{\partial\beta}M_\alpha + \left(\frac{1}{r_\alpha} + \frac{1}{r_\beta}\right)m\right]$$

In addition, when $i = 1$, the quantity $P^{(i)}$ does not become zero in the first two stress-strain relations in (5.6).

Relations (4.4), which are used in the determination of those quantities in (5.1) which bear the additional index 1, also have a simple interpretation. By setting $s = 1$ in the first two equations in (4.4) and taking into account

(5.3), we obtain

$$u_{\gamma, 1}^{(1)} = -\frac{\nu}{2E}(T_1^{(0)} + T_2^{(0)}), \qquad u_{\alpha, 1}^{(1)} = -R\left(\frac{1}{A}\frac{\partial w^{(0)}}{\partial \alpha} + \frac{u^{(0)}}{R_\alpha}\right) \quad (\alpha\beta)$$

The remaining two equations in (4.4) may, with the aid of (5.4), (5.3) and (1.1), be transformed into the form

$$G_1 = -\frac{2Eh^3}{3(1-\nu^2)}\left\{\varkappa_1 + \nu\varkappa_2 - \left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\varepsilon_1^{(0)} - \frac{\nu}{1-\nu}\left(\frac{1}{R_\alpha} + \frac{\nu}{R_\beta}\right)\left(\varepsilon_1^{(0)} + \varepsilon_2^{(0)}\right)\right\} \quad (\alpha\beta) \tag{5.10}$$

$$H_1 = \frac{2Eh^3}{3(1+\nu)}\left(\tau - \frac{\omega^{(0)}}{2R_\alpha}\right), \qquad H_2 = -\frac{2Eh^3}{3(1+\nu)}\left(\tau - \frac{\omega^{(0)}}{2R_\beta}\right)$$

Here, $\varkappa_1$, $\varkappa_2$ and $\tau$ pertain to the curvatures which, as in [3], are given by

$$\varkappa_1 = -\frac{1}{A}\frac{\partial \gamma_1}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}\gamma_2 \quad (\alpha\beta) \tag{5.11}$$

$$\tau = -\frac{1}{A}\frac{\partial \gamma_2}{\partial \alpha} + \frac{1}{AB}\frac{\partial A}{\partial \beta}\gamma_1 + \frac{\omega_2}{R_\alpha}$$

$$\gamma_1 = -\left(\frac{1}{A}\frac{\partial w^{(0)}}{\partial \alpha} + \frac{u^{(0)}}{R_\alpha}\right) \quad (\alpha\beta)$$

$$\omega_1 = \frac{1}{A}\frac{\partial v^{(0)}}{\partial \alpha} - \frac{1}{AB}\frac{\partial A}{\partial \beta}u^{(0)} \quad (\alpha\beta)$$

**6.** The classical equations of the shell theory contain, in addition to terms of the order $h_*^0$, terms of order $h_*^2$, i.e. the equations are formally (but only formally) constructed with an accuracy of order $h_*^2$. In order to bring into correspondence the accuracy of classical theory and that of the currently proposed method and therefrom to proceed to a comparison of results, use will be made of an iterative method of integrating the equations of shell theory which will enable us to check on the attainment of the necessary accuracy.

Neglect in all equations of equilibrium of classical shell theory the terms representing moments of thier derivatives. Then, making use of the last two formulas in (1.7), the fourth and fifth equilibrium equations yield (*)

$$N_1 = -hM_\beta, \qquad N_2 = -hM_\alpha \tag{6.1}$$

Utilizing these formulas to eliminate the transverse stress resultants from the first three equations of equilibrium, we obtain the equilibrium equations of classical membrane theory in which the actual components of exterior surface loading $X$, $Y$ and $Z$ are replaced by the effective components

$$X' = X + h\frac{M_\beta}{R_\alpha} \quad (\alpha\beta), \qquad Z' = Z - \frac{h}{AB}\left[\frac{\partial}{\partial \alpha}(BM_\beta) + \frac{\partial}{\partial \alpha}(AM_\beta)\right] \tag{6.2}$$

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*) The well known equations of shell theory are not written out here. The results given below may be proved by utilizing the equilibrium equations (12.6), part I of monograph [3]. The fifth of these equations has to be corrected by changing the sign in front of $F$ from minus to plus.

The remaining components of classical theory can be made to coincide with relations (5.6), (5.7), (5.10) and (5.11) by eliminating the superscripts in the latter and setting $P^{(i)} = 0$.

The equations of the shell theory which have been simplified by neglecting moments may be integrated step by step in the following manner: the first, second, third and sixth equations of equilibrium yield the tangential stress resultants ($T_1$, $T_2$, $S_1$, $S_2$) ; the tangential strain components $\epsilon_1$, $\epsilon_2$, and $\omega$ may be determined by means of (5.6); the displacement components $u$, $v$ and $w$ may then be determined from (5.7); the curvature components $\varkappa_1$, $\varkappa_2$, $\tau$ are obtained from (5.11); the moments $G_1$, $G_2$, $H_1$ and $H_2$ are determined from (5.10), and the transverse stress resultants $N_1$ and $N_2$ are found from Formulas (6.1). The above method represents a modification (by taking into account moment loads on the surface) of the method described in chapter 5 of monograph [3]. It can be considered as an approximation obtained by an iterative procedure. A second approximation can be constructed by including in the equilibrium equations the quantities $G_1$, $G_2$, $H_1$ and $H_2$ obtained from the initial approximation, etc. However, in order to obtain results with the required accuracy, i.e. to an accuracy of the order of $h_*$ compared with unity, the initial approximation is sufficient.

N o t e . The described iterative method is not always applicable. It can not be used, for example, to find edge effects or the state of stress in shells whose effective length is sufficiently great. The exclusion of such cases from consideration and the pertinent limitations have been discussed in the introduction.

The complete state of stress constructed in this manner will consist of a membrane state of stress produced by tangential forces and a pure moment state of stress produces by moments. The first of these corresponds to that portion of the state of stress discussed in Section 5 which is produced by the stresses

$$R\,h^{-1}\sigma_{\alpha,\,0}^{(0)} + \sigma_{\alpha,\,0}^{(1)}, \quad R\,h^{-1}\tau_{\alpha\beta,\,0}^{(0)} + \tau_{\alpha\beta,\,0}^{(1)}, \quad R\,h^{-1}\sigma_{\beta,\,0}^{(0)} + \sigma_{\beta,\,0}^{(1)} \tag{6.3}$$

The components of the surface loading which produce the state of stress (6.3) may be obtained by combining the components of (5.8) with the components of (5.9) multiplied by $h_*$. With the aid of (1.7), it is easily shown that this procedure yields components with the same accuracy as (6.2). This means that the membrane portions of the states of stress under discussion differ from the state of stress in (6.3) only to the extent that in obtaining (6.3) the dtress-strain relations (5.6) are not homogeneous (in terms of powers of $h_*$ compared to unity). Complete coincidence is obtained if, in the classical theory, the stress-strain relations for $T_1$ and $T_2$ are taken in the form

$$T_1 - \nu T_2 = 2Eh\varepsilon_1 + \nu hm, \qquad T_2 - \nu T_1 = 2Eh\varepsilon_2 + \nu hm \tag{6.4}$$

This correction corresponds to taking into account compression of the shell in the normal direction.

N o t e . The nonhomogeneity in relations (6.4) is easily eliminated by setting

$$T_1 = T_1^* + \frac{\nu h}{1-\nu} m, \qquad T_2 = T_2^* + \frac{\nu h}{1-\nu} m$$

Then the formulas in (6.2) are replaced by the following

$$X'' = X' + \frac{h\nu}{A(1-\nu)} \frac{\partial m}{\partial \alpha} = X + h\,\frac{M_\beta}{R_\alpha} + \frac{h\nu}{A(1-\nu)} \frac{\partial m}{\partial \alpha} \qquad (\alpha\beta)$$

$$Z'' = Z - \frac{h}{AB}\left[\frac{\partial}{\partial \alpha}(BM_\beta) + \frac{\partial}{\partial \beta}(AM_\alpha)\right] + \frac{\nu h}{1-\nu}\left(\frac{1}{R_\alpha} + \frac{1}{R_\beta}\right) m$$

The pure moment part of the state of stress obtained here corresponds to the state of stress $(\zeta\sigma_{\alpha,\,1}{}^{(0)}, \zeta\tau_{\alpha\beta,1}{}^{(0)}, \zeta\sigma_{\beta,\,1}{}^{(0)})$ in Section 5. Coincidence within the bounds of the assumed accuracy will be complete if the relations for the moments are taken in the form (5.10).

From the preceding comparison it follows that, in order to solve the problem formulated in the introduction, we may propose a corrected classical theory the error in which, for $t = 0$, will be of the order $h_*^2$ in comparison with unity. For this, it is necessary: first, to determine the components $X$, $Y$ and $Z$ of the exterior surface loading by Formulas (1.7), retaining only the first two terms in the right-hand sides, i.e. taking into account changes of area scale in going from the exterior or interior surface to the middle surface; second, instead of the actual components of the surface loading, to use the effective components $X'$, $Y'$ and $Z'$, i.e. to take into account moments arising from the transference of exterior tangential forces to the middle surface; third, to make use of the nonhomogeneous relations (6.4) or replace $X'$, $Y'$ and $Z'$ by effective components $X''$, $Y''$ and $Z''$, i.e. to take into account compression arising from the transfer of exterior normal forces to the middle surface; fourth, to take stress-strain relations for the moment resultants in the form (5.10).

7. We will now proceed to study the influence of the index of variation on the errors in determining the state of stress based on classical theory.

We will assume, in (2.2), that $0 < t < \frac{1}{2}$ (here, as in other asymptotic studies, the case where $t \geqslant {}^1/_2$, must be examined separately), and for definiteness, assume ${}^1/_4 < t < {}^1/_3$. This leads to the following inequalities

$$0 < p < q - 2p < 2p < q - p < q < q + p < 2q - 2p \qquad (7.1)$$

which will be used in proving subsequent assertions.

Finally, assume that the quantities in the right hand sides of (1.6) are independent of $h_*$ and that $\sigma_\gamma$ is not identically zero. Then we must set $\rho = 0$ in the series expansions (2.1), and, in order to satisfy the boundary conditions (1.6), it is necessary that

$$\sigma_\gamma^{(s)} = \pm {}^1/_2 Q_\gamma^{(s)} - {}^1/_2 m^{(s)}, \qquad \tau_{\alpha\gamma}^{(s)} = \pm {}^1/_2\, Q_\alpha^{(s)} + {}^1/_2 M_\beta^{(s)} \quad (\alpha\beta) \qquad \text{for } \zeta = \pm 1 \qquad (7.2)$$

where

$$Q_\gamma^{(0)} = Q_\gamma, \quad m^{(0)} = m, \quad Q_\alpha^{(p)} = Q_\alpha \quad (\alpha\beta), \quad M_\alpha^{(p)} = M_\alpha \quad (\alpha\beta) \qquad (7.3)$$

$$Q_\gamma^{(s)} = m^{(s)} = 0 \quad \text{for } s \neq 0, \quad Q_\alpha^{(s)} = M_\beta^{(s)} = Q_\beta^{(s)} = M_\alpha^{(s)} = 0 \quad \text{for } s \neq p$$

The state of stress satisfying all the stated conditions will be referred to as the state of stress $(t > 0,\ Q_\gamma \not\equiv 0)$.

If, in (2.2), the index $s$ satisfies the inequality $s < q - 2p$, then upon taking into account (7.1) and eliminating terms with negative indices it will be found that, when $s = 0$, $p = 0$ and $q = 1$, (2.2) will contain only terms with indices $(s)$ and $(s - p)$. This means that, for $s < q - 2p$, the form of the solution of Equations (2.2) will be the same as that for $s = 0$, $p = 0$ and $q = 1$, i.e. it will be given by Equations (3.2) in which we set $s = 0$.

In particular, this implies that

$$\tau_{\alpha\gamma}^{(s)} = \tau_{\alpha\gamma,0}^{(s)} + \zeta\tau_{\alpha\gamma,1}^{(s)} \qquad (\alpha\beta) \qquad \text{for } s < q - 2p$$

and hence, by virtue of the boundary conditions (7.2) and (7.3),

$$\tau_{\alpha\gamma}^{(s)} = \tau_{\beta\gamma}^{(s)} \equiv 0 \qquad \text{for } s \neq p,\ s < q - 2p$$

If the index $s$ satisfies the inequalities $q - 2p \leqslant s < 2q - 2p$, then (2.2) will contain only terms with the indices $(s)$, $(s - p)$, $(s - q + 2p)$, $(s - q + p)$, $(s - q)$ and $(s - p - q)$, namely those and only those terms which remain in (2.2) when $s = 1$, $p = 0$ and $q = 1$. Thus, the form of the solution is, in this case, given by the Formulas in (3.2) with $s = 1$.

When $s \geqslant 2q - 2p$ then (2.2) will contain terms which do not appear there when $p = 0$ and $q = 1$ until $s \geqslant 2$. Correspondingly, we must take $s > 1$ in (3.2) for such values of $s$.

From the foregoing discussion it follows that if only the first $2q - 2p$ approximations of the state of stress $(t > 0,\ Q_\gamma \not\equiv 0)$ are to be constructed, i.e. if we take $s = 2q - 2p$, then the solution will be of the form

$$\sigma_\alpha = \lambda^q \Big[ \sum_{s=0}^{q-2p-1} \lambda^{-s}\sigma_{\alpha,0}^{(s)} + \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\sigma_{\alpha,0}^{(s)} + \zeta \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\sigma_{\alpha,1}^{(s)} \Big] \qquad (\alpha\beta)$$

$$\tau_{\alpha\beta} = \lambda^q \Big[ \sum_{s=0}^{q-2p-1} \lambda^{-s}\tau_{\alpha\beta,0}^{(s)} + \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\tau_{\alpha\beta,0}^{(s)} + \zeta \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\tau_{\alpha\beta,1}^{(s)} \Big]$$

$$\tau_{\alpha\gamma} = \lambda^p \Big[ \sum_{s=0}^{q-2p-1} \lambda^{-s}\tau_{\alpha\gamma,0}^{(s)} + \zeta \sum_{s=0}^{q-2p-1} \lambda^{-s}\tau_{\alpha\gamma,1}^{(s)} + \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\tau_{\alpha\gamma,0}^{(s)} + \zeta \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\tau_{\alpha\gamma,1}^{(s)} + \zeta^2 \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\tau_{\alpha\gamma,2}^{(s)} \Big] \qquad (\alpha\beta) \tag{7.4}$$

$$\sigma_\gamma = \sum_{s=0}^{q-2p-1} \lambda^{-s}\sigma_{\gamma,0}^{(s)} + \zeta \sum_{s=0}^{q-2p-1} \lambda^{-s}\sigma_{\gamma,1}^{(s)} + \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\sigma_{\gamma,0}^{(s)} + \zeta \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\sigma_{\gamma,1}^{(s)} + \zeta^2 \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}\sigma_{\gamma,2}^{(s)}$$

$$u_\alpha = \lambda^{q-p} \Big[ \sum_{s=0}^{q-2p-1} \lambda^{-s}u_{\alpha,0}^{(s)} + \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}u_{\alpha,0}^{(s)} + \zeta \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}u_{\alpha,1}^{(s)} \Big] \qquad (\alpha\beta)$$

$$u_\gamma = \lambda^q \Big[ \sum_{s=0}^{q-2p-1} \lambda^{-s}u_{\gamma,0}^{(s)} + \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}u_{\gamma,0}^{(s)} + \zeta \sum_{s=q-2p}^{2q-2p-1} \lambda^{-s}u_{\gamma,1}^{(s)} \Big]$$

To obtain the state of stress $(t > 0,\ Q_\gamma \not\equiv 0)$ with errors of the order $\lambda^{-2q+2p} = h_*^{2-2t}$ (here an d hereafter the change from $\lambda$ to $h_*$ is obtained by means of (1.2) and (1.3)),all components in (7.4) must be obtained. This may be accomplished by using only those terms in (2.2) which are used in the computation of the zeroth and first approximations of the state of stress when the index of variation is zero. This leads to :

C o n c l u s i o n 7.1 . The classical shell theory with the corrections formulated at the end of Section 6 enables to determine the state of stress $(t \geq 0,\ Q_\gamma \not\equiv 0)$ with errors of the order $h_*^{2-2t}$.

From (5.3) and (5.4), we can obtain two relations (the first relation is approximate, only the first term having been retained)

$$T_1 \approx 2R\sigma_{\alpha,0}^{(0)}, \qquad G_1 = -\frac{2h^2}{3}\left(\sigma_{\alpha,1}^{(1)} + \frac{\sigma_{\alpha,0}^{(0)}}{r_\beta}\right) \tag{7.5}$$

which are based on (5.1) and are applicable to states of stress with zero index of variation. For states of stress $(t > 0,\ Q_\gamma \not\equiv 0)$ , (5.1) must be replaced by (7.4), so that

$$T_1 \approx 2R\sigma_{\alpha,0}^{(0)}, \qquad G_1 \approx -\frac{2h^2}{3}\left(h_*^{-2t}\sigma_{\alpha,1}^{(q-2p)} + \frac{\sigma_{\alpha,0}^{(0)}}{r_\beta}\right) \tag{7.6}$$

The formulas for the remaining stress resultants and moment resultants may be obtained in a similar manner.

For $t < \frac{1}{2}$ , the major portion of the stresses will be given by the tangential stress resultants. These will be of the order of $h_*^{-1}$ for both states of stress. For $t = 0$ and $t > 0$ , the moments provide corrections of the order $h_*^1$ and $h_*^{1-2t}$, respectively. As one might expect, the importance of the moments increases as $t$ increases. However, it may be seen from the second equations in (7.5) and (7.6) that this behavior is only due to the first terms in the parentheses. The order of magnitude of the second terms does not change as $t$ increases, and consequently, since the terms containing tangential deformations in (5.10) affect only the second terms, the proper formulation is obtained.

C o n c l u s i o n 7.2 . An improper choice of stress-strain relations for the moment resultants leads to a state of stress $(t > 0,\ Q_\gamma \not\equiv 0)$ with errors of order $h_*^1$ regardless of the value of the index of variation $t$ .

Corrections introduced by the proper evaluation of the load components and supplementary terms in (6.4) take effect when $s$ reaches a value such that the terms within braces in (2.2) do not vanish. In view of (7.2) and (7.3), the foregoing condition will occur when $s = q - p$ .

C o n c l u s i o n 7.3 . Errors arising from the improper (in terms of order $h_*$ compared to unity) evaluation of exterior loading lead to errors of the order $\lambda^{q-p} = h_*^{1-t}$ in the determination of the state of stress $(t > 0,\ Q_\gamma \not\equiv 0)$ (here, the errors in the final results are greater than those in the basic equations).

**8.** In Section 7, it was assumed that $Q_\gamma \not\equiv 0$. When $Q_\gamma = m \equiv 0$, i.e. when the shell is carrying only tangential surface loads, then, in order to satisfy the conditions (1.6), we must set $\rho = -p$ in (2.1).

Whereupon (7.2) remains unchanged, and (7.3) is replaced by

$$Q_\gamma^{(0)} = 0 \qquad m^{(0)} = 0, \quad Q_\alpha^{(0)} = Q_\alpha \quad (\alpha\beta), \quad M_\alpha^{(0)} = M_\alpha \quad (\alpha\beta)$$
$$Q_\gamma^{(s)} = m^{(s)} = Q_\alpha^{(s)} = Q_\beta^{(s)} = M_\alpha^{(s)} = M_\beta^{(s)} = 0 \qquad \text{for } s \neq 0$$

All considerations of Section 7 apply, but the terms within braces in the second equation in (2.2) will now become nonzero when $s = q - 2p$. This means that conclusions 7.1 and 7.2 remain in effect, but conclusion 7.3 must be replaced by the following.

C o n c l u s i o n 8.1 . Errors arising from the improper (in terms of order $h_*$ compared to unity) evaluation of exterior loading lead to errors of the order $\lambda^{q-2p} = h_*^{1-2t}$ in the determination of the state of stress $(t > 0, \quad Q_\gamma \equiv 0)$

Thus, in the cases under consideration the largest errors in classical theory arise from inaccuracies in operations connected with the applied load, i.e. inaccuracies which can be eliminated essentially without any difficulty.

**9.** The stress-strain relations obtained here guarantee (when the other previously discussed conditions are satisfied) maximum accuracy in the solution of a certain class of problems described in the introduction. However, these relations are not free of formal contradictions. They do not satisfy the sixth equation of equilibrium, and do not satisfy the necessary conditions for the applicability of the reciprocal theorem (cf. [3], Part I, Section 27). These inconsistencies may be explained in the fact that the formulation is valid only within accuracy of order $h_*$ compared to unity. They may be eliminated without changing the order of accuracy by including in the stress-strain expressions for the tangential stress resultants terms containing bending strain components. This results in the following formulas:

$$T_1 = \frac{2Eh}{1-\nu^2}\left\{\varepsilon_1 + \nu\varepsilon_2 - \left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\varkappa_1 - \frac{\nu}{1-\nu}\left(\frac{1}{R_\alpha} + \frac{\nu}{R_\beta}\right)\left(\varkappa_1 + \varkappa_2\right)\right]\right\} \qquad (9.1)$$

$$S_1 = \frac{Eh}{1+\nu}\,\omega - \frac{Eh^3}{3(1+\nu)}\left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\left(\tau - \frac{\omega}{R_\alpha}\right) H_1 = \frac{2Eh^3}{3(1+\nu)}\left(\tau - \frac{\omega}{2R_\alpha}\right)$$

$$G_1 = -\frac{2Eh^3}{3(1-\nu^2)}\left\{\varkappa_1 + \nu\varkappa_2 - \left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\varepsilon_1 - \frac{\nu}{1-\nu}\left(\frac{1}{R_\alpha} + \frac{\nu}{R_\beta}\right)\left(\varepsilon_1 + \varepsilon_2\right)\right\}$$

(the formulas for $T_2$, $S_2$, $H_2$ and $G_2$, being similar, are not shown; in addition, a term containing $m$ has been left out in the formula for $T_1$).

The stress-strain relations (9.1) are different from those obtained by Lur'e [8] (only the formulas for $H_1$, $H_2$, $S_1$ and $S_2$ coincide). For the class of problems considered here, Formulas (9.1) will certainly yield greater accuracy. For many other problems, both (9.1) and Lur'e's formulas will apparently yield adequate accuracy, but this question requires further examination.

Stress-strain relations (5.6), (5.7), (5.10) and (5.11) correspond to the following assumptions.

1. The stresses $\sigma_\alpha$, $\tau_{\alpha\beta}$, $\sigma_\beta$ and the displacement components $u_\alpha$, $u_\beta$, $u_\gamma$ vary across the shell thickness in the following manner:

$$\sigma_\alpha = \sigma_{\alpha,0} + \gamma\sigma_{\alpha,1} \quad (\alpha\beta), \qquad \tau_{\alpha\beta} = \tau_{\alpha\beta,0} + \gamma\tau_{\alpha\beta,1}$$
$$u_\alpha = u - \gamma\gamma_1 \quad (\alpha\beta), \qquad u_\gamma = -w - \gamma\varphi \tag{9.2}$$

2. The stress-strain relations of three-dimensional theory of elasticity may be given in the form

$$Ee_{\alpha\alpha} = \sigma_\alpha - \nu\sigma_\beta \quad (\alpha\beta), \qquad Ee_{\alpha\beta} = 2(1+\nu)\tau_{\alpha\beta}$$
$$Ee_{\alpha\gamma} = 0 \quad (\alpha\beta), \qquad Ee_{\gamma\gamma} = -\nu(\sigma_\alpha + \sigma_\beta) \tag{9.3}$$

In determining $e_{\alpha\alpha}$, $e_{\beta\beta}$ and $e_{\alpha\beta}$, it is necessary to retain terms containing the zeroth and first powers of $\gamma$, while in the determination of $e_{\alpha\gamma}$, $e_{\beta,\gamma}$ and $e_{\gamma\gamma}$ only terms which are independent of $\gamma$ must be retained.

By expressing $e_{\alpha\alpha}$, $e_{\alpha\beta}$, . . . in terms of $u$, $v$, $w$, $\gamma_1$, $\gamma_2$ and $\varphi$ in the manner described above, then utilizing (9.2) and (9.3) as well as Formulas (5.2) to determine the stress resultants and moment resultants to an accuracy of order $h_*^2$, we obtain the homogeneous (for $P^{(i)} = 0$) stress-strain relations in Section 5. It can be seen, incidentally, that these relations are distinguished from the relations of Lur'e by the fact that here $e_{\gamma\gamma}$ is not taken equal to zero.

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Translated by H.H.